COMPUTATION OF CORRELATIONS IN A LOCALLY ISOTROPIC TURBULENT FLOW

(RASCHET KORRELIATSIONNYKH SVIAZEI V LOKAL'NO IZOTROPNOM TURBULENTNOM POTOKE)

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Various statistical characteristics in the whole interval of equilibrium of locally isotropic turbulence are calculated herein on the basis of a model with a spectral density which attenuates as a Gaussian function for large wave numbers. Where this is possible the characteristics so obtained are compared with results obtained on the basis of other turbulence models: a model with constant skewness and the Heisenberg model in certain places. The structural velocity functions, the longitudinal third moment of the velocity and the skewness coefficient, the structural function and spectrum of the pressure, correlation functions of the vorticity and acceleration are determined. The majority of the obtained characteristics do not depend very strongly on the selection of the turbulence model.

A.N. Kolmogorov's theory of isotropic turbulence explained a number of regularities of turbulent flow at very large Reynolds numbers. In this theory the turbulent flow is considered as a set of vortices of different scales. The very largest vortices, which are characterized by the so-called external scale of turbulence L, originate because of the instability of the whole mean flow. The scale L is of the order of the distance in which the velocity of the mean flow changes significantly. The motion of vortices with scales much smaller than L may be considered homogeneous and isotropic as well as quasi-stationary. Quasi-stationarity means that the change in the statistical characteristics of the motion in the considered range of vortices occurs in a time much greater than the characteristic periods of the given vortices. This interval of scales of motion is characterized by two parameters: the mean dissipation of the vortex kinetic energy per unit mass per unit time $\langle\varepsilon\rangle$ and the kinematic viscosity ν. The angular brackets denote statistical averages. The viscosity in this scale interval is essential only for the very finest vortices for which the intrinsic Reynolds number is of the order of unity. Dissipation is negligible in larger vortices.

A single combination with the dimension of a length $\lambda_0 = (\nu^3/\langle\varepsilon\rangle)^{1/4}$ can be formed from the parameters $\langle\varepsilon\rangle$ and ν. The length λ_0, called the internal scale of turbulence, characterizes the range of vortices in which viscous dissipation of the turbulent kinetic energy occurs. In the scale interval $\lambda_0 \ll r \ll L$ the motion is determined by the single parameter $\langle\varepsilon\rangle$ which measures the intensity of the energy flux transmitted without noticeable loss from the larger to the smaller scale vortices. The influence of viscosity is insignificant here, hence, this scale interval is called inertial. The whole scale interval $r \ll L$ has received the designation of equilibrium interval.

The theory of locally isotropic turbulence, which is limited to the study of the equilibrium interval, requires no knowledge of the average quantities which it is often impossible to determine from experiment. Thus, the determination of the average values for atmospheric turbulence is difficult because of the extremely wide spectrum of the meteorological parameters. Instead of correlation functions for the quantities themselves Kolmogorov's theory introduces the correlation functions of the differences in the quantities being studied, the structural functions. If the average quantities are known then both descriptions become equivalent and the transition from the structural to the correlation functions presents no difficulty.

1. The Kolmogorov theory yields the form of the structural functions of the velocity

$$D_{ij}(r) = \langle [u_i(M) - u_i(M')] [u_j(M) - u_j(M')] \rangle \tag{1.1}$$

in two different scale domains: $r \ll \lambda_0$ and $\lambda_0 \ll r \ll L$, where r is the distance between the observation points M and M'. Because of the homogeneity and isotropy of the turbulence in the equilibrium interval, the structural tensor (1.1) can be represented as

$$D_{ij}(r) = [D_{ll}(r) - D_{nn}(r)]\, r_i r_j / r^2 + D_{nn}(r)\, \delta_{ij} \tag{1.2}$$

where $D_{ll}(r)$ and $D_{nn}(r)$ are the longitudinal and transverse structural functions of the velocity, composed of differences in the longitudinal and transverse components of the velocity. According to Kolmogorov [1, 2]

$$D_{ll}(r) = \frac{1}{15} \frac{\langle\varepsilon\rangle}{\nu} r^2, \qquad D_{nn}(r) = \frac{2}{15} \frac{\langle\varepsilon\rangle}{\nu} r^2 \qquad (r \ll \lambda_0) \tag{1.3}$$

$$D_{ll}(r) = \frac{3}{4} C^2 (\langle\varepsilon\rangle r)^{2/3}, \qquad D_{nn}(r) = C^2 (\langle\varepsilon\rangle r)^{2/3} \qquad (\lambda_0 \ll r \ll L) \tag{1.4}$$

Here C is a constant of order unity, determined from experiment. The exact form of the structural functions of r of order λ_0 is unknown. According to similarity theory, here as in the whole equilibrium interval, the nondimensional functions of the distance should depend only on the ratio r/λ_0. The question of the form of the structural functions for $r \sim \lambda_0$ is equivalent to the determination of the spectrum of velocity fluctuation for large wave numbers $k \gtrsim (\langle\varepsilon\rangle/\nu^3)^{1/4}$. The tensor $D_{ij}(r)$ is related to the spectral tensor by means of the relation

$$D_{ij}(r) = 2\iiint_{-\infty}^{\infty} (1 - e^{i\mathbf{kr}})\, \Phi_{ij}(\mathbf{k})\, d\mathbf{k}, \qquad \Phi_{ij}(\mathbf{k}) = \frac{E(k)}{2\pi k^4}(k^2\delta_{ij} - k_i k_j) \tag{1.5}$$

The tensor $\Phi_{ij}(\mathbf{k})$ is determined by a single scalar function of the spectral density of the turbulent energy $E(k)$.

The first attempt to determine the form of the structural functions of the velocity in the whole equilibrium interval was made by Obukhov [3], who used the hypothesis that the skewness coefficient S of the probability distribution for the longitudinal component of the difference in the velocities at two points in the flow is constant in the whole equilibrium interval

$$S = D_{lll}(r) / [D_{ll}(r)]^{3/2} = \text{const}, \qquad D_{lll}(r) = \langle [v_l(M) - v_l(M')]^3 \rangle \tag{1.6}$$

In the inertial interval as well as for $r \ll \lambda_0$ the skewness coefficient is actually constant [2]. The results of Townsend [4] indicate its approximate constancy even in the intermediate scale interval. However, it has been shown [5] that the structural function of the velocity obtained here seems to possess the requisite asymptotic properties for $r \ll \lambda_0$ and $\lambda_0 \ll r \ll L$ but the corresponding energy spectrum has negative sections in certain domains of wave space. Hence, it is more convenient to look for interpolation formulas for the structural formulas by starting from some sort of positive expressions for the spectrum. Recently Novikov [6] made an attempt to determine the form of the velocity spectrum in the domain of very large wave numbers $k \gg (\langle\varepsilon\rangle/\nu^3)^{1/4}$. He obtained the asymptotic formula

$$E(k) = C_1 \langle\varepsilon\rangle^{2/3} k^{-5/3} (k\lambda_0)^{2m-4/3} \exp[-a(k\lambda_0)^2], \qquad \begin{pmatrix} 0.5 \leqslant m < 1 \\ \sqrt{3} \leqslant a < 2 \end{pmatrix} \tag{1.7}$$

where C_1 is a constant of order unity.

If the additional hypothesis that $m = 2/3$ is made, then (1.7) for $k\lambda_0 \ll 1$ formally transforms into the well known "5/3 law" for the spectrum. Furthermore, if we consider here that formula (1.7) is valid in the whole equilibrium interval, then the parameters a and C_1 can be determined. It turns out that $C_1 = a^{2/3}/\Gamma(2/3)$ and the parameter $a \approx 1.76$. However, in obtaining the latter value, as well as the upper bound for a, the algebraic and averaging operations must be interchanged, which may lead to appreciable error [6].

In this work we shall use a spectral density of the form

$$E(k) = \frac{a^{1/3}}{\Gamma(2/3)} \langle\varepsilon\rangle^{2/3} k^{-5/3} \exp[-a(k\lambda_0)^2] \tag{1.8}$$

as the working model in the whole equilibrium interval, with the justification that this model was given a certain basis in [6]. Such a spectral density had already been used earlier [5].*

The parameter a can be related to the value of the constant C for the structural functions and to the value of the skewness coefficient in the inertial interval. As is known, the relation Kolmogorov [2] established between C and S has the following form

$$C^2 = \frac{4}{3}\left(-\frac{5}{4S}\right)^{2/3} \tag{1.9}$$

Hence, according to the results of paragraph 2

$$a = 2\left[\frac{11\Gamma(11/6)}{12\sqrt{\pi}}\right]^{2/3} C^3 \approx 0.685\, C^3 \tag{1.10}$$

$$a = \frac{64}{15\sqrt{3}}\left[\frac{11\Gamma(11/6)}{12\sqrt{\pi}}\right]^{2/3} \frac{1}{|S|} \approx \frac{0.843}{|S|} \tag{1.11}$$

TABLE

C	S	a	Reference
1.41	0.44	1.92	[2] *
1.46±0.04	0.39±0.03	2.16+0.17	[4]
1.30±0.17	0.55±0.22	1.53±0.61	[8]
1.42±0.09	0.43±0.08	1.96±0.36	[9]
1.58+0.04	0.31+0.02	2.72+0.13	[10]
1.37	0.47	1.76	[6]

* Values of the parameters were calculated by means of the experimental results of [7].

Values of the constant C and the skewness S obtained from experiment exist in a number of references. These data are presented in the table.

Results are presented here of very diverse experiments to measure the turbulence characteristics in wind tunnels [2,4,7], in the atomospheric layer near the earth [8,9], in the ocean [10]. The scatter in the data is rather noticeable and, at present, it is difficult to specify completely the value of the skewness S or the coefficient C. Hence, formula (1.9) should be viewed as a convenient working formula with the free parameter a.

Random fields determined by derivatives of the velocity can be

* The following intuitive foundation is given in this paper: The Navier-Stokes equation is of parabolic type, hence, the velocity pulsations being generated by the large-scale vortices will be smoothed into a fine-scale Gaussian function (or a similar function).

defined, within the scope of the Kolmogorov theory, as simply homogeneous and isotropic random fields in all scales. The mean square and correlation function can be determined for them. A vortex field may serve as a typical example. However, for such fields the theory of locally isotropic turbulence permits the determination of the form of the correlation function only in the inertial interval. The complete determination of the form of the correlation functions requires knowledge of the form of the velocity structural functions in the whole interval of equilibrium.

Random fields defined by the squares of the derivatives of the velocity, such as the field of the Laplacian of the pressure, will also be homogeneous and isotropic fields in all scales. Their correlation functions will be determined by the fourth moments of the velocity, whose relation to the second moments (the structural functions) will require knowledge of the probability distributions of the difference in the velocity components at two points in the flow. The Millionshchikov hypothesis [11], that the fourth moments are expressed in terms of the known second moments exactly as in the normal probability distribution, is usually used here. In this case, from similarity considerations, only the character of the dependence on the distance (the exponent) can be indicated exactly for the correlation functions in the inertial interval. However, since the Millionshchikov hypothesis here introduces not more than a 10-15% error [12], the results obtained by using it are completely satisfactory in many cases. The accuracy of the results obtained by using this hypothesis for scales $r \leqslant \lambda_0$ remains unclear.

Formula (1.8), extended over the whole interval of equilibrium, is the basis of all the subsequent computations. A comparison with results obtained on the basis of a model of turbulence with constant skewness shows that a whole series of turbulent flow characteristics do not depend very strongly on the selection of the model of turbulence.

In certain places, where a great amount of numerical computation would not be required, the Heisenberg model [13] was used in addition to the model of turbulence with constant skewness. In this model, the spectral density

$$E(k) = \left(\frac{8\langle\varepsilon\rangle}{9\gamma}\right)^{2/3} k^{-5/3}\left(1 + \frac{8\nu^3k^4}{3\gamma^2\langle\varepsilon\rangle}\right)^{-4/3} \tag{1.12}$$

decreases comparatively slowly for large k, as k^{-7}. The constant γ can be related to the considered parameter a. Actually, comparing (1.9) and (1.12) in the inertial interval, we find

$$\gamma = \frac{8\Gamma^{3/2}(2/3)}{9a} \approx \frac{1.41}{a} \tag{1.13}$$

The spectral densities (1.8) and (1.12) are, in a certain sense, extreme cases of very rapidly and very slowly decreasing spectra.

Since the parameter a is assumed to be free, it is convenient to represent the results of the computations in a form independent of it. Let us transform to the nondimensional coordinates

$$\varkappa = \sqrt{a}\,\lambda_0 k = \lambda_0' k, \qquad x = r / \lambda_0' \tag{1.14}$$

Let us also introduce the nondimensional structural velocity tensor in the form

$$a\,(\langle\varepsilon\rangle\,\nu)^{1/2}\, d_{ij}\,(r / \lambda_0') = D_{ij}\,(r) \tag{1.15}$$

and the nondimensional spectral density of the kinetic energy as

$$E\,(\varkappa) = \frac{\varkappa^{-5/3}}{\Gamma\,(2/3)} \exp\,(-\varkappa^2) \tag{1.16}$$

2. From (1.5), which relates the structural and spectral tensors, the following formulas [14] for the nondimensional longitudinal and lateral structural functions and the magnitude of the tensor $d_{ij}(x)$ can be obtained by using the isotropy condition

$$d_{ll}\,(x) = 4\int_0^\infty \left(\frac{1}{3} + \frac{\cos\varkappa x}{\varkappa^2 x^2} - \frac{\sin\varkappa x}{\varkappa^3 x^3}\right) E\,(\varkappa)\,d\varkappa \tag{2.1}$$

$$d_{nn}\,(x) = 2\int_0^\infty \left(\frac{2}{3} - \frac{\sin\varkappa x}{\varkappa x} - \frac{\cos\varkappa x}{\varkappa^2 x^2} + \frac{\sin\varkappa x}{\varkappa^3 x^3}\right) E\,(\varkappa)\,d\varkappa \tag{2.2}$$

$$d\,(x) = d_{ll}\,(x) + 2d_{nn}\,(x) = 4\int_0^\infty \left(1 - \frac{\sin\varkappa x}{\varkappa x}\right) E\,(\varkappa)\,d\varkappa \tag{2.3}$$

To compute these functions let us use the well-known integrals [15]

$$\int_0^\infty k^\lambda e^{-k^2} \cos kydk = \frac{1}{2}\,\Gamma\left(\frac{\lambda+1}{2}\right) M\left(\frac{\lambda+1}{2}, \frac{1}{2}, -\frac{y^2}{4}\right) \tag{2.4}$$

$$\int_0^\infty k^{\lambda-1} e^{-k^2} \sin kydk = \frac{1}{2}\,\Gamma\left(\frac{\lambda+1}{2}\right) yM\left(\frac{\lambda+1}{2}, \frac{3}{2}, -\frac{y^2}{4}\right) \tag{2.5}$$

where $M(\alpha,\ \gamma,\ z)$ is the confluent hypergeometric function. Using the recurrence formulas relating the confluent hypergeometric functions which differ by an integer in the values of the parameters α and γ (e.g. see [16] and the appendix as well), it is possible to obtain the formulas

$$d_{ll}\,(x) = 2\left[M\left(-\frac{1}{3},\ \frac{5}{2},\ -\frac{x^2}{4}\right) - 1\right] \tag{2.6}$$

$$d_{nn}(x) = 3M\left(-\frac{1}{3},\ \frac{3}{2},\ -\frac{x^2}{4}\right) - M\left(-\frac{1}{3},\ \frac{5}{2},\ -\frac{x^2}{4}\right) - 2 \quad (2.7)$$

$$d(x) = 6\left[M\left(-\frac{1}{3},\ \frac{3}{2},\ -\frac{x^2}{4}\right) - 1\right] \quad (2.8)$$

Hence, for $x \ll 1$, using the known expansions of the function $M(\alpha, \gamma, z)$, we have

$$d_{ll}(x) = \frac{x^2}{15} - \frac{x^4}{630} + \ldots,\ d_{nn}(x) = \frac{2x^2}{15} - \frac{x^4}{210} + \ldots,\ d(x) = \frac{x^2}{3} - \frac{x^4}{90} + \ldots \quad (2.9)$$

It is evident that the principal terms of the expansion agree with formulas (1.3).

According to [16], for $x \gg 1$

$$d_{ll}(x) = \frac{9\sqrt{\pi}\,(x/2)^{2/3}}{11\Gamma(11/6)}\left(1 + \frac{22}{9x^2} - \ldots\right) - 2 \approx 0.967x^{2/3}\left(1 + \frac{22}{9x^2} - \ldots\right) - 2 \quad (2.10)$$

$$d_{nn}(x) = \frac{12\sqrt{\pi}\,(x/2)^{2/3}}{11\Gamma(11/6)}\left(1 + \frac{11}{18x^2} - \ldots\right) - 2 \approx 1.29x^{2/3}\left(1 + \frac{11}{18x^2} - \ldots\right) - 2 \quad (2.11)$$

$$d(x) = \frac{3\sqrt{\pi}\,(x/2)^{2/3}}{\Gamma(11/6)}\left(1 + \frac{11}{9x^2} - \ldots\right) - 6 \approx 3.56x^{2/3}\left(1 + \frac{11}{9x^2} - \ldots\right) - 6 \quad (2.12)$$

In the given model, the presence of a constant component in the asymptote is characteristic for large distances as it is not, for example, in the model with constant asymmetry [3,17].

Equating formulas (2.11) and (1.4), we obtain (1.10); hence, we obtain (1.11) by using (1.9). The longitudinal and lateral structural functions are pictured by solid lines on Fig. 1. The method of computing the hypergeometric functions is given in the appendix. The structural functions corresponding to the hypothesis of constant skewness [3,17] are shown by dashed lines. Because of the relation between the parameter a and the skewness, a conversion of the results of [3,17] for comparison with ours is obtained by a simple scale transformation

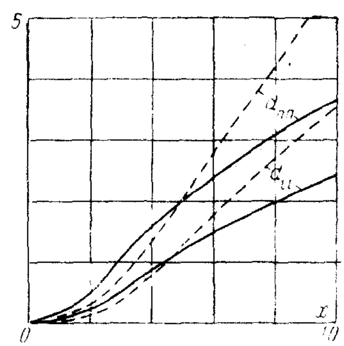

Fig. 1.

$$x = 5.48x_s, \qquad d_{ij}(x) = 4.08d_{ij}^s(x_s)$$

The principal terms of the expansions for $x \ll 1$ and $x \gg 1$ will be the same for the structural functions in both models, however, the asymptote of the structural functions for $x \gg 1$ will be reached outside the limits of this figure for S = const in the given scale. For $x \gg 1$ the structural functions of the two models will differ by a constant equal to 2.

In the Heisenberg model an analytic form for the structural functions has not been obtained successfully in the whole interval of equilibrium, hence, let us be limited to finding the second term of the expansion, at zero, of the structural functions (evidently the first term will be common for all isotropic models). To do this, let us expand the parentheses in the integrands in (2.1) and (2.2) in Taylor series up to terms of order x^4 inclusively.

Integrating the obtained expressions with $E(k)$ in the form (1.12), we obtain in the variables under consideration

$$d_{ll}(x) = \frac{x^2}{15} - \frac{\sqrt{\pi}\,\Gamma(11/6)\,\Gamma^{3/2}(5/3)}{2\,100\Gamma(4/3)}x^4 + \ldots = \frac{x^2}{15} - 0.000763\,x^4 + \ldots$$

$$d_{nn}(x) = \frac{2x^2}{15} - \frac{\sqrt{\pi}\,\Gamma(11/6)\,\Gamma^{3/2}(5/3)}{700\Gamma(4/3)}x^4 + \ldots = \frac{2x^2}{15} - 0.00229x^4 + \ldots$$

3. Kolmogorov's equation [2] permits the computation of the third moment of the velocity pulsation if the second moment is known. In the nondimensional variables of (1.12) and (1.14), this equation takes the form

$$ad_{lll}(x) = -\tfrac{4}{5}x + 6d_{ll}'(x) \qquad (3.1)$$

where the nondimensional third moment is introduced by using the relation

$$D_{lll}(r) = a^{3/2}(\langle\varepsilon\rangle)^{3/4} d_{lll}(r/\lambda_0') \qquad (3.2)$$

From (3.1) and (2.6), we obtain

$$ad_{lll}(x) = -\frac{4}{5}x\left[1 - M\left(\frac{2}{3}, \frac{7}{2}, -\frac{x^2}{4}\right)\right] \qquad (3.3)$$

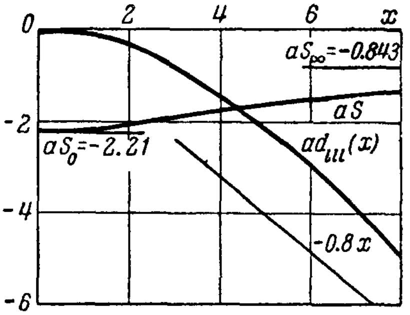

Fig. 2.

The asymptotic expansions have the following form:

$$ad_{lll}(x) = -\frac{4x^3}{105} + \frac{2x^5}{567} - \cdots \qquad (x \ll 1) \qquad (3.4)$$

$$ad_{lll}(x) = -\frac{4x}{5} + \frac{9\sqrt{\pi}\cdot 2^{2/3}}{11\Gamma(11/6)}x^{-1/3} - \ldots = -0.8x + 3.38x^{-1/3} - \ldots \qquad (x \gg 1) \qquad (3.5)$$

Hence, it is seen that the longitudinal third moment approaches its asymptotic value $-0.8x$ rather slowly. The results of calculations of the third moment are given in Fig. 2.

From (3.3) and (2.6) we obtain the following expression for the skewness:

$$S(r) = \frac{D_{lll}(r)}{[D_{ll}(r)]^{3/2}} = -\frac{0.8x\,[1 - M(2/3,\ 7/2,\ -x^2/4)]}{2^{3/2}a\,[M(-1/3,\ 5/2,\ -x^2/4) - 1]^{3/2}} \tag{3.6}$$

Hence, the following expansions can be obtained:

$$aS(x) = -\frac{4\sqrt{15}}{7}\left(1 - \frac{43x^2}{756} + \cdots\right) = 2.21\,(1 - 0.057x^2 + \ldots) \qquad (x \ll 1) \tag{3.7}$$

$$aS(x) = -0.843\,(1 + 3.10x^{-2/3} - 4.85x^{-4/3} + \ldots) \qquad (x \gg 1) \tag{3.8}$$

The behavior of the skewness (more accurately, the quantities $aS(x)$) is also shown in Fig. 2. Let us note that for the case under consideration the skewness at zero is 2.6 times greater than in the inertial interval.

Values of the skewness at zero for the Heisenberg, Kovasznay and Obukhov models are presented in Reid's paper [18]. In our variables, these quantities equal, respectively, 2.14, 7.20, 3.62. Values of the skewness in the Heisenberg model are very similar to the value obtained in the model with a spectrum decaying as a Gaussian function. The two last values are noticeably greater than the two first. This latter circumstance is perhaps due to the fact that the Kovasznay and Obukhov spectra vanish beyond a certain wave number.

In concluding this section let us examine one circumstance for which Reid found no explanation, when he compared the value of the skewness at zero with the value in the expression for the spectrum in the inertial interval. Let us recall that skewness in the Kolmogorov theory is defined by (1.7) and it is precisely this expression for $\lambda_0 \ll r \ll L$ which enters into the formula for the spectrum in the inertial interval.

The skewness at zero is defined as

$$S_0 = \left\langle \left(\frac{\partial u}{\partial x_1}\right)^3 \right\rangle \left\langle \left(\frac{\partial u}{\partial x_1}\right)^2 \right\rangle^{-3/2}$$

It is easy to see that (1.7) transforms into this latter only if $r \to 0$. The Kolmogorov skewness should not be the same in the inertial interval as near zero.

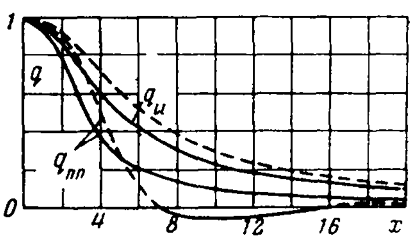

Fig. 3.

4. A computation of the vorticity correlation functions can be made by the appropriate differentiation of the structural functions of the velocity. As is known, the correlation component of the vortex vector $\omega = \text{rot}\ \mathbf{v}$ is expressed in terms of the correlation of the velocities as follows [13]:

$$\langle\omega_i(M)\,\omega_j(M')\rangle = Q_{ij}(r) = -\,\delta_{ij}\Delta R_{kk}(r) + \frac{\partial^2 R_{kk}(r)}{\partial r_i \partial r_j} + \Delta R_{ij}(r) \quad (4.1)$$

where $R_{ij}(r)$ is the correlation tensor of the velocities. Hence, the following expressions for the longitudinal and lateral vorticity correlation functions can be obtained in the terminology of the velocity structural functions:

$$Q_{ll}(r) = -\frac{1}{2}\frac{\partial^2 D_{ll}(r)}{\partial r^2} + \frac{2}{r}\frac{\partial D_{nn}(r)}{\partial r} \quad (4.2)$$

$$Q_{nn}(r) = \frac{1}{2}\left[\frac{\partial^2 D_{ll}(r)}{\partial r^2} + \frac{\partial^2 D_{nn}(r)}{\partial r^2} + \frac{1}{r}\frac{\partial D_{ll}(r)}{\partial r}\right] \quad (4.3)$$

Using (1.3) and (1.4), it is possible to determine the behavior of the vorticity correlation functions near zero and in the inertial interval. For $r \ll \lambda_0$ we obtain

$$Q_{ll} = \frac{7}{15}\frac{\langle\varepsilon\rangle}{\nu}, \qquad Q_{nn} = \frac{4}{15}\frac{\langle\varepsilon\rangle}{\nu}, \qquad Q_{kk} = Q_{ll} + 2Q_{nn} = \frac{\langle\varepsilon\rangle}{\nu} \quad (4.4)$$

In the inertial interval

$$Q_{ll} = \tfrac{17}{12}C^2\langle\varepsilon\rangle^{2/3}r^{-4/3}, \qquad Q_{nn} = \tfrac{1}{18}C^2\langle\varepsilon\rangle^{2/3}r^{-4/3}, \qquad Q_{kk} = \tfrac{55}{36}C^2\langle\varepsilon\rangle^{2/3}r^{-4/3} \quad (4.5)$$

Using the results of computations performed for [5], it is possible to construct normalized vorticity correlation functions q_{ll} and q_{nn} for the model of turbulence with constant skewness. These functions are given dashed in Fig. 3 on the scale used there. The correlation functions for a model with the spectral density (1.8) are shown by the continuous line. The following formulas can be obtained for these functions (4.6)

$$q_{ll}(x) = M\left(\frac{2}{3}, \frac{5}{2}, -\frac{x^2}{4}\right), \quad q_{nn}(x) = \frac{3}{2}M\left(\frac{2}{3}, \frac{3}{2}, -\frac{x^2}{4}\right) - \frac{1}{2}M\left(\frac{2}{3}, \frac{5}{2}, -\frac{x^2}{4}\right)$$

The asymptotic expansions here are

$$q_{ll}(x) = 1 - 0.0688x^2 + \ldots, \qquad q_{nn}(x) = 1 - 0.133x^2 + \ldots \qquad (x \ll 1) \quad (4.7)$$

$$q_{ll}(x) = 3.78x^{-4/3}(1 - 2.22x^{-2} + \ldots), \quad q_{nn}(x) = 1.26x^{-4/3}(1 + 4.34x^{-2} - \ldots) \quad (x \gg 1)$$

The difference between the lateral correlation function $q_{nn}(x)$ and an analogous function found under the hypothesis of constant skewness , strikes the eye. The latter has a negative section while the former is positive everywhere. In completely homogeneous turbulence q_{nn} must actually have a negative section, as follows from the vector ω being solenoidal. Actually, the equality

$$q_{nn}(x) = q_{ll}(x) + \frac{1}{2} x q_{ll}'(x) = \frac{1}{2x}(x^2 q_{ll})' \tag{4.8}$$

results from the condition div $\omega = 0$.

Hence, integrating by parts and assuming a sufficiently rapid decay of $q_{ll}(x)$ at infinity, it is possible to obtain the relation

$$\int_0^\infty x^m q_{nn}(x)\,dx = \frac{1-m}{2}\int_0^\infty x^m q_{ll}(x)\,dx \tag{4.9}$$

For $m = 1$

$$\int_0^\infty x q_{nn}(x)\,dx = 0 \tag{4.10}$$

i.e. the function q_{nn} must have negative sections. However, it is necessary for compliance with condition (4.1) that the function $q_{ll}(x)$ should decrease more rapidly than x^{-2}. Here, this function is calculated only in the inertial interval where the decrease occurs according to the law $x^{-4/3}$ so that the equality (4.1) does not hold in the present case. Hence, it is a coincidence that the lateral correlation function has a negative section under the hypothesis that the skewness is constant. An analogous remark can be made about the longitudinal correlation function of the acceleration also (see Section 7 below), evaluated in [17] under the same hypothesis S = const, which also has a negative section. The decay in the correlation functions there is weaker still and occurs as $x^{-2/3}$. The negative sections, where they exist, should be observed in scales larger than the external scale of turbulence, i.e. beyond the inertial interval where the correlation apparently actually decreases sufficiently rapidly.

5. Obukhov and Iaglom [17] gave a method of computing the pressure structural function

$$\Pi(r) = \langle [p(M) - p(M')]^2 \rangle \tag{5.1}$$

Here, the following equation is used

$$\Delta p = -\rho \frac{\partial v_i}{\partial x_k}\frac{\partial v_k}{\partial x_i} \tag{5.2}$$

which is obtained from the Navier-Stokes equation and the incompressibility condition.

Multiplying (5.2), taken at the point M, by an analogous equation at the point M', using the homogeneity condition and the Millionshchikov hypothesis, it is possible to obtain an equation for the function $\Pi(r)$

$$\Delta^2\Pi(r) = \frac{d^4\Pi}{dr^4} + \frac{4}{r}\frac{d^3\Pi}{dr^3} = -\rho^2\frac{\partial^2 D_{ik}}{\partial r_j \partial r_l}\frac{\partial^2 D_{jl}}{\partial r_i \partial r_k} = -\rho^2\Phi(r) \qquad (5.3)$$

Let us introduce the nondimensional functions $\pi(x)$ and $\varphi(x)$

$$\Pi(r) = a^2\rho^2\langle\varepsilon\rangle\nu\pi(r/\lambda_0'), \qquad \Phi(r) = a^2\langle\varepsilon\rangle\nu^{-2}\varphi(r/\lambda_0') \qquad (5.4)$$

Then, according to [17]

$$\pi(x) = \frac{1}{6}\int_0^x\left(-3z^2 + 3xz + \frac{1}{x}z^3\right)\varphi(z)\,dz + \frac{x^2}{6}\int_x^\infty z\varphi(z)\,dz \qquad (5.5)$$

The function $\varphi(x)$ can be written as

$$\varphi(x) = \frac{6}{x^2}d_{ll}'^2(x) + \frac{4}{x}d_{ll}'(x)d''(x) + 4d_{ll}''^2(x) \qquad (5.6)$$

In the case under consideration calculations yield

$$\varphi(x) = \tfrac{8}{3}\left(\tfrac{7}{15}M_{7/2}^2 + \tfrac{2}{3}M_{5/2}^2 + \tfrac{2}{5}M_{3/2}M_{7/2} - \tfrac{4}{3}M_{5/2}M_{7/2}\right) \qquad (5.7)$$

Here and later, in the interests of brevity, we shall use the notation

$$M_\gamma = M\left(\frac{2}{3},\ \gamma,\ -\frac{x^2}{4}\right) \qquad (5.8)$$

The function $\varphi(x)$, used repeatedly later, is actually the correlation of the Laplacians of the pressure. Its asymptotes are

$$\varphi(x) = \tfrac{8}{15}\left(1 - \tfrac{2}{9}x^2 + \ldots\right) \qquad (x \ll 1) \qquad (5.9)$$

$$\varphi(x) = \frac{7\pi\cdot 2^{5/3}}{121\Gamma^2(11/6)}x^{-8/3} \approx 0.652x^{-8/3} \qquad (x \gg 1) \qquad (5.10)$$

Hence, the function $\varphi(x)$ diminishes rather rapidly but remains always positive. The normalized correlation function of the Laplacians of the pressure $\varphi_1(x) = (15/8)\varphi(x)$ is pictured in Fig. 4.

The nondimensional structural function of the pressure $\pi(x)$ must be proportional to x^2 for $x \ll 1$ because of the smoothing effect of the viscosity; from (5.5) and (5.9), we have the proportionality coefficient

$$b = \frac{1}{6}\int_0^\infty z\varphi(z)\,dz \qquad (5.11)$$

This integral was evaluated numerically and equals 0.294. In order to calculate the next terms in the expansion of $\pi(x)$ it is necessary to use the asymptote of $\varphi(x)$ for $x \ll 1$. Using (5.10), we obtain

$$\pi(x) = 0.294x^2 - \frac{1}{225}x^4 + \ldots \qquad (5.12)$$

Let us note that the second term of the expansion of the function $\Pi(r)$ is independent of the turbulence model within the scope of the Millionshchikov hypothesis for $r \ll \lambda_0$ and is determined only by the local isotropy of the flow.

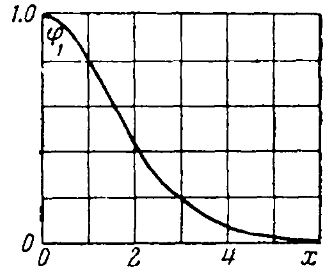

Fig. 4.

Knowledge of the first term in the expansion of $\pi(x)$ for small x permits the finding of the mean-square of the gradient of the turbulent pressure pulsations

$$\langle(\nabla p)^2\rangle = \frac{1}{2}\frac{\partial^2 \Pi(0)}{\partial r_i^2} = \frac{1}{2}\Delta\Pi(0) \qquad (5.13)$$

Calculations yield

$$\langle(\nabla p)^2\rangle = 0.882a\rho^2\langle\varepsilon\rangle^{3/2}\nu^{-1/2} = B\rho^2\langle\varepsilon\rangle^{3/2}\nu^{-1/2} \qquad (5.14)$$

The value of the coefficient B in [19] for the mean square of the pressure gradient is $1.1/|S|$ or, if (1.11) is taken into account, then $B = 1.3a$. Batchelor [20], using a function of the form

$$D_{ll}(y) = \frac{y^2/15}{[1 + (15C_b)^{-3/2}y^2]^{2/3}} \qquad \left(C_b = \frac{3}{4}C^2\right)$$

also estimated this coefficient.

In his computation Batchelor assumed $C_b = 2.0$ ($C = 1.63$) and he found that his coefficient $B = 1.3$. It is not difficult to see that the coefficient B contains C_b as the factor $C_b^{3/2}$, i.e. after certain calculations his result can be represented in the form $B = 0.31\,C^3 = 1.36a$. According to [21], $B = 1.4\,\gamma^{-1} = 1.0a$ for the Heisenberg spectrum. Hence, four different turbulence models yield quantities for the mean square of the pressure gradient which differ within a range of the order of 50 per cent.

Let us return to the calculation of the pressure structural function for large values of x. Here, as Obukhov [22] showed, the function $\Pi(r) = \rho^2 D_{ll}^2(r)$ or, according to (2.12), in this case $\pi(x) = 0.940x^{4/3}$. But, as was detected in [17], there is a term proportional to x in the expansion of $\pi(x)$ for large values of the argument. However, it is more convenient at the beginning to compute the function $\pi(x)$ numerically for intermediate x and then, by using these results, to determine the asymptote for $x \gg 1$. The numerical integration was carried out for the range $0.4 \leqslant x \leqslant 8$. In calculating the asymptote for $x \gg 1$ it is necessary to take into account that the first integral in (5.5) diverges as x increases. Hence, it was broken into two parts: from 0 to 8, where it was integrated numerically, and from 8 to ∞, where it was integrated by using the asymptotic expansion of the function $\varphi(x)$. After addition of these two parts and the execution of other necessary operations, we obtain

$$\pi(x) = 0.940x^{4/3} - 0.107x - 2.35 + 4.33x^{-2/3} - 1.83x^{-4/3} + \ldots \qquad (5.15)$$

The pressure structural function is pictured in Fig. 5. The structural function computed in [17] under the hypothesis of constant skewness, is superposed by dashes for comparison: the asymptotes for this function are somewhat different.

In the scale used there they have the following form:

$$\pi(x) = 0.415x^2, \quad (x \ll 1), \qquad \pi(x) = 0.940x^{4/3} - 3.08x + 14.4 \quad (x \gg 1) \qquad (5.16)$$

6. For a number of applications it is necessary to know the spectrum of the pressure fluctuations in a turbulent flow. In particular, if the pressure spectrum is multiplied by k^4, then up to a numerical factor, we obtain the spectrum of the fluctuation of the dissipation [23]. In calculating the pressure spectrum it is convenient to start from the equation

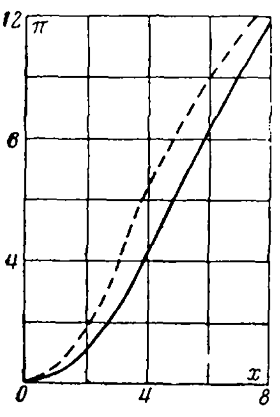

Fig. 5.

$$\langle \Delta p(M)\, \Delta p(M') \rangle = \Delta^2 (\langle pp' \rangle) = = \frac{1}{2}\rho^2 \frac{\partial^2 D_{ik}}{\partial r_j \partial r_l} \frac{\partial^2 D_{jl}}{\partial r_i \partial r_k} = \frac{1}{2}\rho^2 \Phi(r) \qquad (6.1)$$

obtained from (5.3). Executing a Fourier transformation on this equation and averaging over the angles, we obtain the formula

$$k^4 F_p(k) = \frac{\rho^2}{4\pi^2 k} \int_0^\infty r\Phi(r) \sin kr \, dr \qquad (6.2)$$

where $F_p(k)$ is the three-dimensional spectral density of the pressure. Let us introduce the nondimensional function $g(\kappa)$

$$4\pi^2 k^5 F_p(k) = a^3 \rho^2 \langle \varepsilon \rangle^{3/2} \nu^{-1/2} g(k\lambda_0') \qquad (6.3)$$

Then it follows from (6.2) that

$$g(\varkappa) = \int_0^\infty x\varphi(x)\sin \varkappa x\, dx \qquad (6.4)$$

By using this expression it is not difficult to find the asymptote of the function $g(\kappa)$ for $\kappa \ll 1$, i.e. in the inertial interval. It is easy to see that this asymptote is determined by the behavior of $\varphi(x)$ for $x \gg 1$. Simple computations yield

$$g(\varkappa) = \frac{63\pi\sqrt{3}\,\Gamma(4/3)}{2^{1/3}121\,\Gamma^2(11/6)}\varkappa^{2/3} \approx 2.26\varkappa^{2/3} \qquad (6.5)$$

It is convenient to start from another representation, which can be obtained from (6.1) by using the convolution theorem [20], when evaluating the pressure spectrum for large wave numbers

$$F_p(k) = \frac{1}{8\pi^2}\iiint_{-\infty}^{\infty} E(|\mathbf{k}'|)\,E(|\mathbf{k}-\mathbf{k}'|)\,\frac{\sin^4\theta}{|\mathbf{k}-\mathbf{k}'|^4}\,d\mathbf{k}' \qquad (6.6)$$

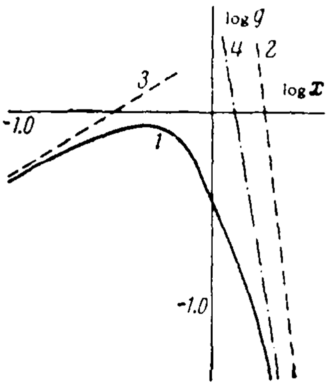

Fig. 6.

Here $E(|\mathbf{k}|)$ is the spectral density of the velocity pulsations; θ the angle between the vectors $\mathbf{k}$ and $\mathbf{k}'$. Transforming to nondimensional variables, introducing spherical coordinates and integrating over the angle φ we obtain when (1.8) is taken into account

$$F_p'(\varkappa) = \frac{1}{4\pi\Gamma^2(2/3)}\int_0^\pi \sin^5\theta\, d\theta \int_0^\infty \frac{\varkappa'^{1/3}\exp[-\varkappa'^2 - (\varkappa^2 - 2\varkappa\varkappa'\cos\theta + \varkappa'^2)]}{(\varkappa^2 - 2\varkappa\varkappa'\cos\theta + \varkappa'^2)^{17/6}}\, d\varkappa' \qquad (6.7)$$

The asymptotic expansion of this integral for $\kappa \gg 1$ equals

$$F_p'(\varkappa) = \frac{256\cdot 2^{5/6}(1+\operatorname{erf} 1)}{9e^2\Gamma^2(5/3)}\varkappa^{-34/3}e^{-1/2\varkappa^2}(1 + O(\varkappa^{-2})) \approx 15.5\varkappa^{-34/3}e^{-1/2\varkappa^2} \qquad (6.8)$$

For the value $\kappa \sim 1$ the spectrum was calculated by numerical integration of (6.4). The function $g(\kappa)$, which is proportional to the pressure spectrum, is pictured in Fig. 6. The asymptote (6.5) and the asymptote corresponding to (6.8)

$$g(\varkappa) = 612\varkappa^{-10/3}e^{-1/2\varkappa^2} \qquad (6.9)$$

are given by dashes.

As a comparison with these results, let us calculate the asymptote of the pressure spectrum in the Heisenberg model for large wave numbers.

Substituting the spectral density (1.12) into (6.7), we obtain after calculations

$$F_p'(\varkappa) = \frac{27\Gamma^3(5/3)}{40\pi}\varkappa^{-11}(1+O(\varkappa^{-1})) \approx 0.158\varkappa^{-11}, \quad g(\varkappa) = 4\pi^2\varkappa^5 F_p'(\varkappa) \approx 6.24\varkappa^{-6} \quad (6.10)$$

The last asymptote is superposed in Fig. 6 by a dash-dot line.

7. The statistical configuration of the acceleration field of fluid particles in a turbulent flow was studied in detail in [17]. Let us use the fundamental results of the computation performed there for the case under consideration and let us determine the mean square of the acceleration and the correlation functions. As a result of the equations of motion the acceleration components of the fluid particles equal

$$w_i = \frac{dv_i}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\Delta v_i$$

It is possible to obtain the formula

$$\langle w_i^2\rangle = \frac{1}{2\rho^2}\Delta\Pi(0) - \frac{\nu^2}{2}\Delta^2 D(0) \quad (7.1)$$

for the mean square acceleration.

Using (5.9) and (2.11), we obtain

$$\langle w_i^2\rangle = (0.882a + 2/3a)\langle\varepsilon\rangle^{3/2}\nu^{-1/2} \quad (7.2)$$

Theoretically $a \geqslant \sqrt{3}$ for the asymptotic formula (1.8) (as is generally justified by experiment (see the table)), hence, the mean-square acceleration in a turbulent flow is determined mainly by the fluctuating pressure gradients. Iaglom [19] obtained the following expression under the hypothesis S = const:

$$\langle w_i^2\rangle = \left(\frac{1.1}{|S|} + 0.3|S|\right)\langle\varepsilon\rangle^{3/2}\nu^{-1/2} = \left(1.3a + \frac{0.25}{a}\right)\langle\varepsilon\rangle^{3/2}\nu^{-1/2} \quad (7.3)$$

It is characteristic that frinctional forces play a relatively large part in (7.2) than in (7.3). Thus, for $|S|$ = 0.47 (a = 1.76) the viscous friction forces are about 20 per cent and about 5 per cent, respectively, of the totals in the mean-square acceleration. Estimates of the magnitudes of the acceleration for the turbulence model under consideration are somewhat reduced under the different conditions presented in [17, 19] but the general deductions on the possibilities of large values of turbulent accelerations remain valid.

Let us compute the acceleration correlation functions taking account of the effect of viscosity since it can be notable as compared with the case of the model of turbulence with constant skewness [17] where the correlation of the pressure gradients was computed in practice. However, the parameter a is not successfully eliminated from the equation for the acceleration correlation functions; it is not invariant with respect to a as is seen from the example of the mean-square acceleration (7.2). Hence, let us compute separately the correlation of the pressure gradients and the correlation due to viscous forces. The sum of these two will be the total acceleration correlation.

The following expressions were obtained in [17] for the longitudinal and lateral correlation functions of the acceleration

$$A_{ll}(r) = \frac{1}{2\rho^2}\Pi''(r) + \nu^2\left(\frac{4}{r^3}D_{ll}' - \frac{4}{r^2}D_{ll}'' - \frac{4}{r}D_{ll}''' + \frac{1}{2}D_{ll}^{IV}\right) = A_{ll}^p + A_{ll}^\nu \quad (7.4)$$

$$A_{nn}(r) = \frac{1}{2\rho^2 r}\Pi'(r) - \quad (7.5)$$
$$- \nu^2\left(\frac{2}{r^3}D_{ll}' - \frac{2}{r^2}D_{ll}'' + \frac{4}{r}D_{ll}''' + \frac{3}{4}D_{ll}^{IV} + \frac{1}{4}D^{IV}\right) = A_{nn}^p + A_{nn}^\nu$$

Let us first calculate the correlation due to the effect of viscosity. Let us introduce the nondimensional functions

$$\alpha_{ll}^\nu(r/\lambda_0') = a\nu^{1/2}\langle\varepsilon\rangle^{-3/2}A_{ll}^\nu(r), \qquad \alpha_{nn}^\nu(r/\lambda_0') = a\nu^{1/2}\langle\varepsilon\rangle^{-3/2}A_{nn}^\nu(r) \quad (7.6)$$

The functions $\alpha_{ll}^\nu(x)$ and $\alpha_{nn}^\nu(x)$ can be obtained by differentiating (2.6) and (2.8). Using the recursion relations for the confluent hypergeometric functions, we obtain the following formulas (see the notation in (5.8)):

$$\alpha_{ll}^\nu(x) = \frac{1}{2}M_{3/2} - \frac{5}{18}M_{5/2}, \qquad \alpha_{nn}^\nu(x) = \frac{1}{4}M_{1/2} - \frac{1}{6}M_{3/2} + \frac{5}{36}M_{5/2} \quad (7.7)$$

For large x

$$\alpha_{ll}^\nu(x) = \frac{5\cdot 2^{1/3}\sqrt{\pi}}{9\Gamma(11/6)}x^{-10/3} \approx 1.48x^{-10/3}, \quad \alpha_{nn}^\nu(x) = \frac{5\cdot 2^{1/3}\sqrt{\pi}}{3\Gamma(11/6)}x^{-10/3} \approx 4.44x^{-10/3} \quad (7.8)$$

To compute the correlation of the pressure gradients let us introduce the functions

$$\alpha_{ll}^p(r/\lambda_0') = a^{-1}\nu^{1/2}\langle\varepsilon\rangle^{-3/2}A_{ll}^p(r), \qquad \alpha_{nn}^p(r/\lambda_0') = a^{-1}\nu^{1/2}\langle\varepsilon\rangle^{-3/2}A_{nn}^p(r) \quad (7.9)$$

According to [17]

$$\alpha_{ll}^p(x) = \frac{1}{2}\pi''(x) = \frac{1}{6x^3}\int_0^x z^4\varphi(z)\,dz + \frac{1}{6}\int_x^\infty z\varphi(z)\,dz$$

$$\alpha_{nn}^p(x) = \frac{1}{2x}\pi'(x) = \frac{1}{4}\int_0^x\left(z^2 - \frac{z^4}{x^2}\right)\varphi(z)\,dz + \frac{1}{6}\int_x^\infty z\varphi(z)\,dz$$

where the function $\varphi(z)$ is determined by equalities (5.6) and (5.7).

Using the properties of the functions $\pi(x)$ and $\varphi(x)$, we can obtain the expansions

$$\alpha_{ll}^{p}(x)=0.294-\frac{2}{75}x^{2}+\ldots,\quad \alpha_{nn}^{p}(x)=0.294-\frac{2}{225}x^{2}+\ldots \qquad (x\ll 1) \quad (7.10)$$

$$\alpha_{ll}^{p}(x)\approx 0.164x^{-2/3},\qquad \alpha_{nn}^{p}(x)\approx 0.493\ x^{-2/3} \qquad (x\gg 1) \quad (7.11)$$

Hence, the correlation of the pressure gradients diminishes much more slowly than the correlation due to viscosity. The latter can only play a part at small distances.

A numerical integration was carried out for intermediate values of x. The normalized correlation functions of the pressure gradients are pictured in Fig. 7. For comparison, corresponding correlation functions, computed in [17] under the hypothesis of constant asymmetry, are superposed by dashes. The different character of the behavior of the longitudinal correlation functions was mentioned at the end of Section 4.

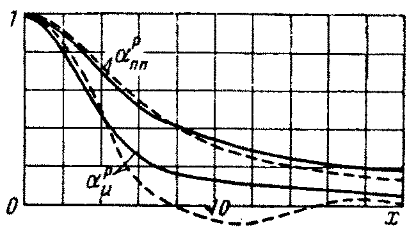

Fig. 7.

The statistical configuration of the random field of kinetic energy dissipation can be studied in an analogous manner, as was done by the author earlier [22].

Appendix. Functions which are expressed in terms of confluent hypergeometric functions $M(\alpha,\ \gamma,\ -1/4x^{2})$ with different values of the parameters α and γ, correspond to a spectrum which decreases as $\exp(-k^{2})$ for large wave numbers. The function $M(\alpha,\ \gamma,\ z)$ is defined by an infinite series which converges for any value of z. However, for $z \geqslant 1$ this series converges slowly. An asymptotic expansion [16] can be used for $z \gg 1$. The function $M(\alpha,\ \gamma,\ z)$ is not tabulated for the values of the parameters α and γ encountered herein, hence, special calculations were carried out. Formulas (2.4) and (2.5) were used for this, namely, the integrals

$$\int_{0}^{\infty} k^{1/3} e^{-k^{2}} \cos kx dk=\frac{1}{2}\Gamma\left(\frac{2}{3}\right) M\left(\frac{2}{3}, \frac{1}{2}, -\frac{x^{2}}{4}\right) \qquad (1)$$

$$\int_{0}^{\infty} k^{-2/3} e^{-k^{2}} \sin kx dk=\frac{1}{2} x \Gamma\left(\frac{2}{3}\right) M\left(\frac{2}{3}, \frac{3}{2}, -\frac{x^{2}}{4}\right) \qquad (2)$$

were evaluated numerically for values of x from 1.0 to 4.0. Appropriate series were used for lesser and greater values of the argument.

All the remaining, required functions $M(\alpha, \gamma, -1/4x^2)$ differ from the computed two by an integer in the values of the parameters α and γ and can be found by means of recursion relations connecting such functions [16]. Let us note that the last of the recursion formulas in all editions of [16] has a misprint. It should be

$$z(\gamma-\alpha)M(\alpha,\gamma+1,z)=\gamma(z+\gamma-1)M(\alpha,\gamma,z)+\gamma(\gamma-1)M(\alpha,\gamma-1,z) \qquad (3)$$

Let us also present one more recursion relation which is very often useful in computations

$$\alpha zM(\alpha+1,\gamma+2,z)=\gamma(\gamma+1)[M(\alpha,\gamma),z)-M(\alpha,\gamma+1,z)] \qquad (4)$$

which is verified by direct substitution of the series for the confluent hypergeometric function.

In conclusion, I am grateful to A.M. Obukhov for discussions and comments during performance of the research and also to A.S. Monin and A.M. Iaglom who read the manuscript and made a number of remarks.

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